Local and log BPS invariants

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- We count curves in X in class $\beta \in H_2(X, \mathbb{Z})$.
- $E \in |-K_S|$ a smooth elliptic curve.

S: del Pezzo surface

$X = \mathrm{Tot}(K_S)$
local del Pezzo

Compare?

$X = (S, E)$
log pair

Local (BPS) invariant

Log (BPS) invariant

Talk 1 : Local BPS invariants

Physicists' interpretation ('98)

$$M \xrightarrow{p} V_\beta \xrightarrow{q} pt$$

- M is the "space" of pairs (C, L).
- C : curve in class β, L : line bundle of degree $p_a(\beta)$.
- V_β is the space of curves in class β.
- p, q are forgetting maps.
- $\mathcal{H}$: a cohomology theory on M.

Physicists' claim

- Lefschetz actions relative to p and q define an $sl_2 \times sl_2$-action on $\mathcal{H}$.
- To count genus g curve on X, we "count" cohomology of genus g Jacobian in $\mathcal{H} \Rightarrow$ get a number n_β^g the BPS invariant.

It is not clear at all!

Example. $S = \mathbb{P}^2$, $\beta = 3H$.

$$\begin{array}{c} M \\ \downarrow p \\ \mathbb{P}^9 \\ \downarrow q \\ pt \end{array}$$

- M is space of (C, p), C a cubic curve and $p \in C$.
- $\mathbb{P}^9$ is the space of cubic curves.
- M is isomorphic to a $\mathbb{P}^8$-bundle over $\mathbb{P}^2$.

$$M \simeq \mathbb{P}(\mathcal{K}), \quad \mathrm{rank}(\mathcal{K}) = 9$$

- $H^*(M) = \mathbb{C}[H, \xi]/(H^3, \xi^9 - 3\xi^8 H + 9\xi^7 H^2)$.
  $H = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{K})}(1))$.

But, we cannot find the expected $sl_2 \times sl_2$-representation structure on $H^*(M)$.

What is M?

In physics, n_β^g is the "count of D-branes".
In mathematics, D-branes are sheaves!

- $M_\beta = \{F\colon F \text{ stable sheaf on } X \text{ with } [F] = \beta, \chi(F) = 1\}$.
  F is stable w.r.t. $L(= -K_S)$ if
  1. F is pure (no 0-dim subsheaves)
  2. For a proper $G \subset F$, $\frac{\chi(G)}{r(G)} < \frac{\chi(F)}{r(F)}$. where $r(F) = L \cdot [F]$ is the linear coefficient of the Hilbert polynomial.
- On a smooth curve (having class β), such sheaves are degree g line bundles, where g is the genus of the curve.
- M_β is equipped with a symmetric obstruction theory.
- Roughly, BPS invariants n_β^g count the genus g Jacobian in M_β.

Mathematicians' rigorous answer ('12)

Kiem-Li's beautiful masterpiece arXiv:1212.6444.

$$\begin{array}{c} M_\beta \\ \downarrow p \\ V_\beta \\ \downarrow q \\ pt \end{array}$$

- M_β is a critical virtual manifold (locally looks like a critical locus of a hol. fcn.).
- There is a global perverse shaef P on M.
- Locally, P is a perverse sheaf of vanishing cycle.
- $\chi(\mathbb{H}^*(M,P))$ recovers the virtual invariant.
- $\sum_t t^i \dim \mathbb{H}^i(M,P)$ is the refined DT invariant.
- $\mathbb{H}^*(M,P)$ has relative Hard Lefschetz.
- Hard Lefschetz on p (q) defines the left (right) sl_2-action.

Therefore, $\mathbb{H}^*(M,P)$ is $sl_2 \times sl_2$-representation as we wanted.
Cor. $n^0_\beta = DT(M_\beta) := \deg[M_\beta]^{\mathrm{vir}}$.

sl_2-representation

- sl_2 is three dimensional Lie algebra generated by $\{h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$.
- For each $m \geq 0$, there is an irreducible representation V_m of sl_2 of dimension $m+1$.

$$V_m := \langle v_0, \cdots, v_m \rangle$$
$$h(v_j) \in \langle v_j \rangle, e(v_j) \in \langle v_{j-1} \rangle, f(v_j) \in \langle v_{j+1} \rangle,$$

- V_m is denoted by $[\frac{m}{2}]$.
- For example, the cohomology ring of the torus T^2

$$H^*(T^2, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$$

has the Lefschetz decomposition and it is as sl_2-representation

$$[\frac{1}{2}] \oplus 2[0]$$

Gopakumar-Vafa (BPS) invariant

Let $j_L, j_R \in \frac{1}{2}\mathbb{Z}$.
$[j_L, j_R]$ is an irreducible $sl_2 \times sl_2$-representation.

$$\mathbb{H}^*(M, P) = \sum_{j_L, j_R} N^{\beta}_{j_L, j_R} [j_L, j_R]$$

- $N^{\beta}_{j_L, j_R}$ is the refined BPS index.

Consider the decomposition w.r.t. the left action.

$$\begin{aligned} Tr(-1)^{F_R}\mathbb{H}^*(M, P) &= \sum_{j_L} N^{\beta}_{j_L, j_R}(-1)^{2j_R}(2j_R + 1)[j_L] \\ &= \sum_{g} n^g_{\beta}([\frac{1}{2}] \oplus 2[0])^{\otimes g}. \end{aligned}$$

- n^g_{β} is the Gopakumar-Vafa (BPS) invariant.

Gromov-Witten invariants

- $\overline{M}_g(X,\beta)$= moduli space of stable maps
  $$= \{f : C \to X, f_*[C] = \beta, |Aut(f)| < \infty\}$$
- proper Delign-Mumford stack with perfect obstruction theory. ([Li-Tian, Behrend-Fantechi])
- virtual cycle $[\overline{M}_g(X,\beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(\overline{M}_g(X,\beta))$
  $$\mathrm{vdim} = (\dim X - 3)(1-g) - \int_\beta \omega_X$$
- for a CY 3-fold X, the GW invariant for X is
  $$\mathcal{I}_\beta^g(X) = \deg[\overline{M}_g(X,\beta)]^{\mathrm{vir}}.$$

Multiple cover formula

$\mathcal{I}^g_\beta(X)$ is not an actual count of curves.

Aspinwall-Morrison multiple cover formula

Let $C \subset X$ be a rigid smooth rational curve.
The contribution of degree k covers of C to $\mathcal{I}^0_\beta(X)$ is $\frac{1}{k^3}$.

Conjecture (Katz, Hosono-Saito-Takahashi)

When genus is zero,

$$\mathcal{I}^0_\beta(X) = \sum_{k|\beta} \frac{n^0_{\beta/k}}{k^3}$$

Proved for local del Pezzo surfaces by [Toda], [Maulik-Nekrasov-Okounkov-Pandharipande] and others.

GW-BPS correspondence

In general,

Conjecture (Gopakumar-Vafa)

$$\sum_{g,\beta} \mathcal{I}^g_\beta(X) q^\beta \lambda^{2g-2} = \sum_{g,\beta,k} n^g_\beta \frac{1}{k}(2\sin(\frac{k\lambda}{2}))^{2g-2} q^{k\beta}.$$

This conjecture is widely open.

Setup

- S: a del Pezzo surface. (A surface with ample $-K_S$)
- S is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $S_r :=$ blowup of $\mathbb{P}^2$ at r points $(0 \leq r \leq 8)$.
- $\beta \in H_2(S, \mathbb{Z})$ a curve class. We often consider β as a divisor on S.
- $w = (-K_S) \cdot \beta$.
- $p_a(\beta) = \frac{1}{2}\beta(\beta + K) + 1$.
- Fix $\mathcal{O}_S(1) = -K_S$.
- We consider genus zero invariants. (We will omit the superscript g.)

Local BPS invariants

- A stable sheaf on $X = \mathrm{Tot}(K_S)$ must be supported on S.
- M_β = Moduli space of stable sheaves F on S with $[F] = \beta$ and $\chi(F) = 1$.
- (Le Potier) M_β is a smooth projective variety of dimension $\beta^2 + 1$. So,

$$n_\beta = (-1)^{\beta^2+1}\chi(M_\beta)$$

Examples

For $S = \mathbb{P}^2$ and $\beta = dH$.

1. $M_1 = \{\mathcal{O}_L\} \simeq \mathbb{P}^2$. L : line. $n_1 = 3$
2. $M_2 = \{\mathcal{O}_C\} \simeq \mathbb{P}^5$. C : conic. $n_2 = -6$.
3. $M_3 = \{\mathcal{F} \text{ such that } 0 \to \mathcal{O}_C \to \mathcal{F} \to \mathcal{O}_p \to 0\}$,
   C : cubic, $p \in C$
   $=$ the universal cubic curve
   $\simeq \mathbb{P}^8$-bundle on $\mathbb{P}^2$. $n_3 = 27$.
4. $\chi(M_4) = 192$. ([Sahin, C-Chung, C-Maican]) $n_4 = -192$.
5. $\chi(M_5) = 1695$. ([C-Chung, Maican]) $n_5 = 1695$.
6. $\chi(M_6) = 17064$. ([C-Chung]) $n_6 = -17064$.

Local BPS numbers for local $\mathbb{P}^2$

d	$p_a(d)$	n_d	$(-1)^{3d-1}n_d/(3d)$
1	0	3	1
2	0	-6	1
3	1	27	3
4	3	-192	16
5	6	1,695	113
6	10	-17,064	948
7	15	188,454	8,974
8	21	-2,228,160	92,840
9	28	27,748,899	1,027,737
10	36	-360,012,150	12,000,405

Our conjecture

- Observation : n_d for local $\mathbb{P}^2$ is divisible by $3d$.
- $3d = (-K_{\mathbb{P}^2}) \cdot \beta$.

Conjecture

For local del Pezzo surface,

- n_β *is divisible by* $w := (-K_S) \cdot \beta$.
- $(-1)^{w-1} n_\beta / w$ *coincides with the log BPS invariant* m_β.

We proved Conjecture for all β with $p_a(\beta) \leq 2$.

Log BPS invariants

- The log BPS invariant m_β roughly count the curves in class β having maximal tangency with a smooth anticanonical curve E at a certain point P.
- $\mathcal{N}_\beta^P(S,E)$ is the corresponding GW-type invariants.

$$\mathcal{N}_\beta^P(S,E) = \sum_{\{k|\beta\,:\,P\in E(\beta/k)\}} \frac{(-1)^{(k-1)w/k}}{k^2} m^P_{\beta/k}.$$

- More details on Talk 2!

Local BPS computation

Theorem

Let S be a del Pezzo suface and β be a curve class on S. Assume there are at least one irreducible curves in class β. Let h be the number of (-1)-curves on S not intersecting β.

- *If $p_a(\beta) = 0$, then $n_\beta = (-1)^{w-1} w$.*
- *If $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $n_\beta = (-1)^{w-1} w(\chi(S) - h)$.*
- *If $\beta = -K_{S_8}$, then $n_\beta = (-1)^{w-1} w(\chi(S_8) + 1) = 12$.*
- *If $p_a(\beta) = 2$, then $n_\beta = (-1)^{w-1} w(\binom{\chi(S)-h}{2} + 5)$.*

Sketch of proof

It is enough to consider only very ample β's.

Lemma (di Rocco)

With a few exceptions, a curve class β is very ample if and only if $\beta \cdot \ell > 0$ for all (-1)-curves ℓ.

Lemma

Let $\pi : S' \to S$ be a blowup of a point. Then $M_{\pi^*\beta}(S') \simeq M_\beta(S)$. So,

$$n_{\pi^*\beta}(S') = n_\beta(S).$$

$\Longrightarrow$ By blowing down all (-1)-curves ℓ with $\beta \cdot \ell = 0$, it is enough to consider very ample β's.

For very ample β's, we use PT-BPS correspondence (KKV method).

Pandharipande-Thomas (PT) stable pair invariant

- Let X be a Calabi-Yau threefold and $\beta \in H_2(X)$.
- A PT stable pair (F, s) on X consists of a sheaf F on X with a section $s \in H^0(F)$ such that
  - F is pure of dimension 1 (No zero-dimensional subsheaf).
  - s generates F outside of finitely many points.
- The moduli space $P_n(X, \beta)$ of stable pairs with $[F] = \beta$, $\chi(F) = n$ can be constructed by GIT. (Le Potier)
- $P_n(X, \beta)$ admits a symmetric obstruction theory.

$$\Longrightarrow \text{ Pandharipande-Thomas invariant } PT_{n,\beta}.$$

PT-BPS Correspondence

- $Z_{PT} = \sum_{n,\beta} PT_{n,\beta} q^n t^\beta$.
- Conjecture(PT-BPS correspondence) :

$$Z_{PT} = \prod_{\beta} \left(\prod_{j=1}^{\infty} \left(1 - (-q)^{j+1} t^\beta\right)^{j n_\beta^0} \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} \left(1 - (-q)^{g-k} t^\beta\right)^{(-1)^{k+g} n_\beta^g \binom{2g-2}{k}} \right),$$

where n_β^g is the BPS invariant.

Katz-Klemm-Vafa method (cf. [C.-Katz-Klemm])

There is a procedure to compute all n_β^g (or more generally all refined BPS indices). In particular,

$$n_\beta^0 = PT_{1,\beta} - PT_{-1,\beta} + \text{correction terms},$$

where correction terms are from reducible curves.

Example

For $S = \mathbb{P}^2$ and $\beta = 4H$. $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-3))$.

- $P_1(X, 4H) = \{(C, Z) \mid \deg C = 4,\ Z \subset C,\ \text{length } Z = 3\}$.
- $P_1(X, 4H)$ is a $\mathbb{P}^{11}$-bundle over $Hilb^3(\mathbb{P}^2)$.
$$\chi(P_1(X, 4H)) = 12 \times 22 = 264, PT_{1,4H} = -264.$$
- Similarly, $P_{-1}(X, 4)$ is a $\mathbb{P}^{13}$-bundle over $Hilb^1(\mathbb{P}^2)$.
$$\chi(P_{-1}(X, 4H)) = 14 \times 3 = 42, PT_{1,4H} = -42.$$
- The correction term is $-PT_{0,3H} \cdot PT_{1,H} = -(-10) \cdot 3 = 30$.
- Therefore,
$$n_{4H} = -264 - (-42) + 30 = -192.$$

δ-stable pairs

- A pair on X is a pair (F, s) of a coherent sheaf F on X and a nonzero section $s \in H^0(F)$.
- Let δ be a positive rational number, $r(F)$ be the leading coefficient of $\chi(F(m))$. A pair (F, s) is called δ-(semi)stable if
  1. F is pure (no zero dimensional subsheaf).
  2. For all proper nonzero subsheaf F' of F, we have

$$\frac{\chi(F') + \epsilon(s, F')\delta}{r(F')} < (\leq) \frac{\chi(F) + \delta}{r(F)},$$

  where $\epsilon(s, F') = 1$ if s factors through F' and 0 otherwise.
- $M_n^\delta(X, \beta)$: Moduli space of δ-semistable pairs (F, s) on X such that $[F] = \beta$ and $\chi(F) = n$ (Le Potier).

$$\frac{\chi(F') + \epsilon(s, F')\delta}{r(F')} < (\leq) \frac{\chi(F) + \delta}{r(F)},$$

Assume β and χ are coprime.

- For a sufficiently large $\delta := \infty$,

$$M_n^{\infty}(X, \beta) = P_n(X, \beta).$$

- For a sufficiently small $\delta := 0^+$, we have the forgetting map

$$M_1^{0^+}(X, \beta) \to M_\beta.$$

$$
\begin{array}{ccc}
PT_1(X,\beta) & \dashleftarrow\dashrightarrow & M_1^{0^+}(X,\beta) \\
 & & \downarrow \\
 & & M_\beta
\end{array}
$$

- Then we have $\chi(M_\beta) = \chi(M_1^{0^+}(X,\beta)) - \chi(M_{-1}^{0^+}(X,\beta))$.
- Compare with

  $n_\beta = PT_{1,\beta} - PT_{-1,\beta} +$ correction terms,
- The correction term are from the wall-crossing between $M_n^\infty(X,\beta)$ and $M_n^{0^+}(X,\beta)$

Wall-crossing

- The moduli space remains unchanged except for only finitely many δ. (Such δ is called the walls.)
- δ is a wall if and only if there exists strictly semistable objects.
- Let $0 \to A \to P \to B \to 0$ be a destabilizing sequence at a wall. Then P is stable on the one side of the wall, but not stable on the other side, where it is replaced by new objects $0 \to B \to P' \to A \to 0$. (Simple wall-crossing)

BPS numbers for local del Pezzo

- 0-very ample=globally generated, 1-very ample = very ample...

Lemma

Let $g = p_a(\beta)$. If β is $(g+n-2)$-very ample, then $P_n(X,\beta)$ is a $\mathbb{P}^{w-n}$-bundle over $Hilb^{g+n-1}(S)$.

- When β is very ample and $p_a(\beta) \leq 2$, the relavant PT spaces are smooth.
- For $p_a(\beta) = 0$ or 1, there are no wall-crossing contributions.
- For non-very ample β with $p_a(\beta) = 1$, blowing down corresponds to wall-crossing. (cf. compare with log BPS calculation.)
- For $p_a(\beta) = 2$, we computed the wall-crossing and calculated n_β.

Remarks

- If $p_a(\beta) \geq 2$, the computation is harder.
  - The PT moduli space is not smooth. The PT invariants are not known in general.
  - The wall-crossing is not of simple type. But we may apply Joyce-Song type virtual wall-crossing.
  - In principle, BPS numbers can be obtained by mirror symmetry and B-model.
- The same calculation applies to the Poincaré polynomials (the refined invariants) of M_β.
  - Conj. The Poincaré polynomial of M_β is also divisible by $(1 + t + \cdots + t^{w-1})$.
  - Q. Is there a corresponding refinement for log BPS invariants?

BREAK

Coming up next:

Talk 2 : Log BPS invariants

Preparation

Joint work with J. Choi, S. Katz and N. Takahashi.

Setup

- $S =$ del Pezzo surface.
- $\beta \in \mathrm{H}_2(S, \mathbb{Z})$ curve class.
- $w = \beta \cdot (-K_S)$ its degree.
- $E =$ smooth effective anticanonical divisor (an elliptic curve).

Goal

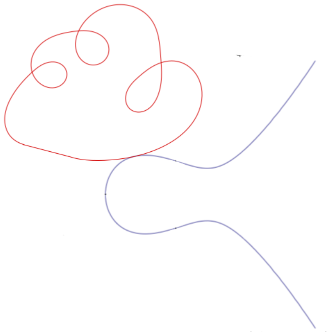
- Enumerative interpretation for the n_β (conjectural).
- Connection to quiver DT invariants.
- More practice with moduli spaces.

Local Gromov-Witten invariants of S

- Recall that for the local CY 3-fold $X = \mathrm{Tot}(K_S)$, the genus 0 degree β GW invariant is
$$\mathcal{I}_\beta(X) = \deg[\overline{M}_0(X, \beta)]^{\mathrm{vir}}.$$
- They are virtually counting genus 0 stable maps $f : C \to \mathrm{Tot}(K_S)$ of degree $f_*[C] = \beta$.

Relative g=0 GW invariants of (S, E) of maximal tangency

$\mathcal{N}_\beta(S, E) := \#^{vir}$ of g=0 relative stable maps $C \to S$ meeting E in exactly one point of maximal tangency. Generic image curve:



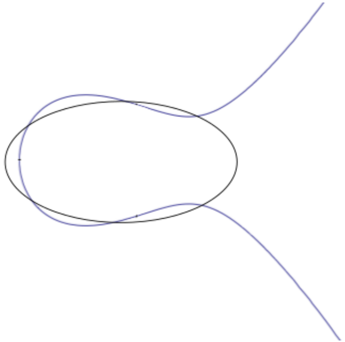
The theory (moduli space and obstruction theory) was developed by J. Li. For a generalization, see Gross-Siebert.

Think of

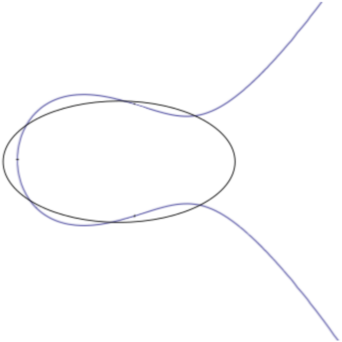
E = boundary. We counts maps $\mathbb{A}^1 \to S - E$ extending to $\mathbb{P}^1 \to S$, meeting E in one point.

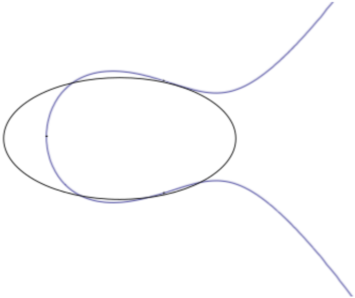
Virtual dimension = 0 by example

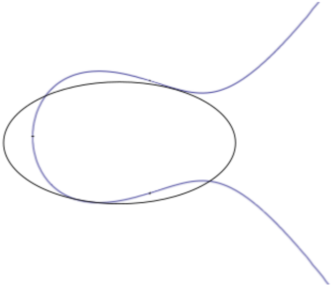
The space of conics in $\mathbb{P}^2$ has dimension 5. A generic conic meets E in 6 points:

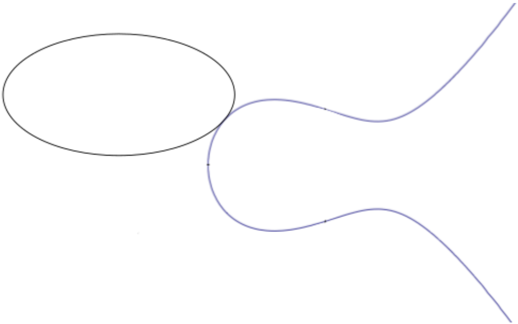


Each time 2 intersection points are moved together, intersection multiplicity at that point $+1$ and vdim -1.







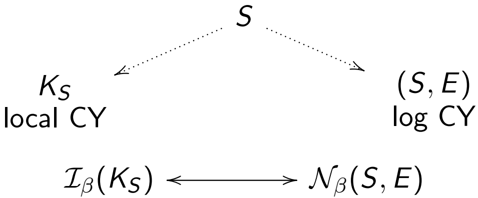


In the end, after 5 identifications, vdim = 0.

A local-relative correspondence theorem

Theorem (Gathmann, Graber-Hassett, van Garrel-Graber-Ruddat)

$$(-1)^{w-1} w\, \mathcal{I}_\beta(K_S) = \mathcal{N}_\beta(S,E).$$



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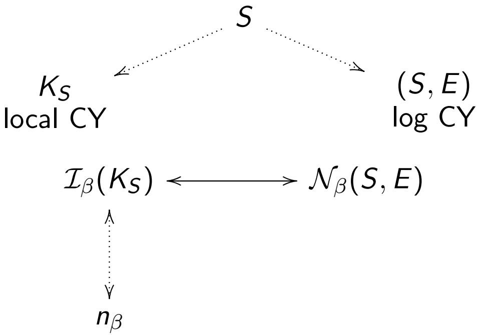

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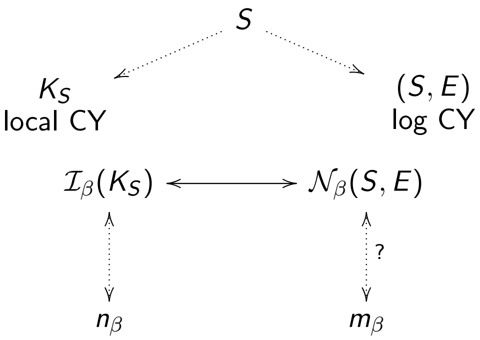


Why relative GW?

2 Major Advantages

- Finite number of possible image curves in each degree.
- Interplay with the arithmetic of E.

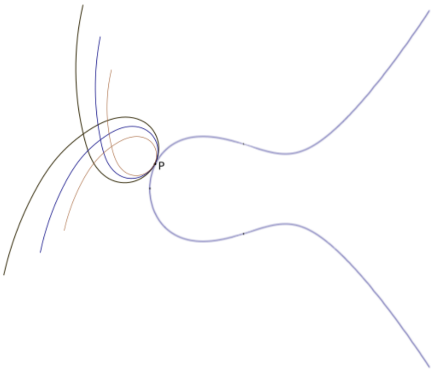
$$E(\beta) := \{P \in E \ : \ \exists C \text{ of degree } \beta \text{ s.t. } C|_E = wP\}\,, \text{ finite set.}$$

The relevant moduli space $\overline{\mathcal{M}}_\beta(S,E)$ decomposes as

$$\overline{\mathcal{M}}_\beta(S,E) = \coprod_{P \in E(\beta)} \overline{\mathcal{M}}_\beta^P(S,E), \text{ and hence}$$
$$\mathcal{N}_\beta(S,E) = \sum_{P \in E(\beta)} \mathcal{N}_\beta^P(S,E).$$

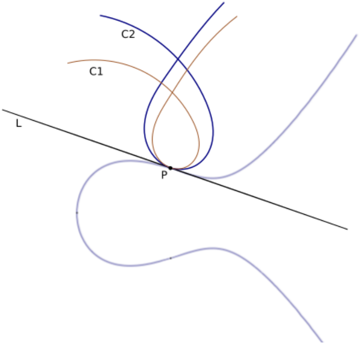
Degree 3 over primitive P in $\mathbb{P}^2$

- $\overline{\mathcal{M}}^P_{3H}(\mathbb{P}^2, E) = 3$ isolated points.
- $\mathcal{N}^P_{3H}(\mathbb{P}^2, E) = 3$.



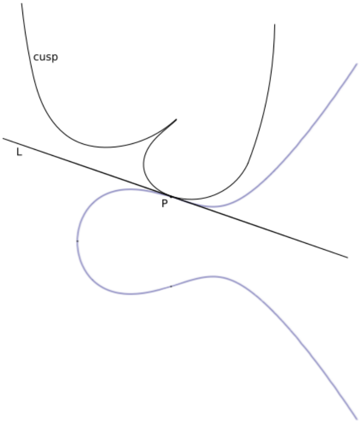
Degree 3 over a flex P - generic E

- $\overline{\mathcal{M}}^P_{3H}(\mathbb{P}^2, E) = [\{C1\}] \sqcup [\{C2\}] \sqcup \overline{\mathcal{R}}$, where $\overline{\mathcal{R}} = \{3:1 \text{ relative covers } C \to L\}$ is 2-dimensional.
- $\mathcal{N}^P_{3H}(\mathbb{P}^2, E) = 1 + 1 + \frac{10}{9} \neq 3$ (Gross-Pandharipande-Siebert).



Degree 3 over a flex P - special E

- $\overline{\mathcal{M}}^{P}_{3H}(\mathbb{P}^2, E) = [\{cusp\}^2] \sqcup \overline{\mathcal{R}}$.
- $\mathcal{N}^{P}_{3H}(\mathbb{P}^2, E) = 2 + \frac{10}{9} = 1 + 1 + \frac{10}{9}$.



Recall: $\sum_{k|\beta} \frac{n_{\beta/k}}{k^3} = \mathcal{I}_\beta(X)$ and $(-1)^{w-1} w \, \mathcal{I}_\beta(K_S) = \mathcal{N}_\beta(S,E)$.

Identities, after a suggestion of P. Bousseau

$$(-1)^{w-1} w \, n_\beta = \sum_{k|\beta} \frac{(-1)^{(k-1)w/k}}{k^2} \mu(k) \underbrace{\mathcal{N}_{\beta/k}(S,E)}_{||}$$

$$= \sum_{k|\beta} \frac{(-1)^{(k-1)w/k}}{k^2} \mu(k) \sum_{P \in E(\beta/k)} \mathcal{N}^P_{\beta/k}(S,E)$$

$$= \sum_{P \in E(\beta)} m^P_\beta, \quad \text{where:}$$

Definition, after P. Bousseau

$$m^P_\beta := \sum_{\{k|\beta \,:\, P \in E(\beta/k)\}} \frac{(-1)^{(k-1)w/k}}{k^2} \mu(k) \mathcal{N}^P_{\beta/k}(S,E).$$

Definition

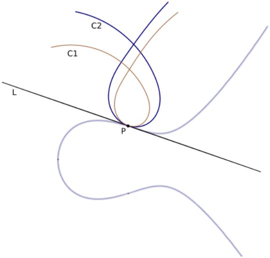
$$m_\beta^P := \sum_{\{k|\beta\,:\,P\in E(\beta/k)\}} \frac{(-1)^{(k-1)w/k}}{k^2}\mu(k)\,\mathcal{N}_{\beta/k}^P(S,E).$$

Equivalently,

$$\mathcal{N}_\beta^P(S,E) = \sum_{\{k|\beta\,:\,P\in E(\beta/k)\}} \frac{(-1)^{(k-1)w/k}}{k^2} m_{\beta/k}^P.$$

Degree 3 over a flex P - generic case, revisited.

- For Q non-flex, $\mathcal{N}^{Q}_{3H}(\mathbb{P}^2, E) = 3 = m^{Q}_{3H}$.
- For P a flex, $\mathcal{N}^{P}_{3H}(\mathbb{P}^2, E) = 2 + \frac{10}{9}$.
- $m^{P}_{3H} = 1 \cdot N^{P}_{3H}(\mathbb{P}^2, E) - \frac{1}{9} \cdot N^{P}_{H}(\mathbb{P}^2, E) = 2 + \frac{10}{9} - \frac{1}{9} \cdot 1 = 3.$



It follows from [N. Takahashi, unpublished] that the m^{P}_{β} agree for $\mathbb{P}^2$ in degree ≤ 4, leading to:

Conjecture A (very hard)

$$\forall P, P' \in E(\beta),\ m_\beta^P = m_\beta^{P'}.$$

Since $|E(\beta)| = w^2$, Conj. A $\implies$

$$(-1)^{w-1} w\, n_\beta = \sum_{P \in E(\beta)} m_\beta^P = w^2\, m_\beta^Q \text{ for any } Q \in E(\beta).$$

Conjecture B (weaker)

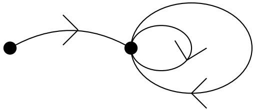
Choose $P \in E(\beta)$ to be β-primitive (definition later). Then

$$(-1)^{w-1} \frac{n_\beta}{w} = m_\beta^P.$$

N. Takahashi '96 implies that Conj. B holds for $\mathbb{P}^2$ and degree ≤ 8.

Important Remarks

- $\mathcal{N}^P_\beta(S,E)$ is further decomposed and m^P_β further refined according to the disjoint components of $\overline{\mathcal{M}}^P_\beta(S,E)$.
- Adjusted multiple cover contributions of components of $\overline{\mathcal{M}}^P_\beta(S,E)$ yield quiver DT invariant contributions to m^P_β.
- In the previous example, $m^P_{3H} = 2 + \frac{10}{9} - \frac{1}{9} \cdot 1 = 3$,
  $\mathbb{N} \ni \frac{10}{9} - \frac{1}{9} = DT^2_3$ is the 2-loop quiver DT invariant of degree 3:



More generally

- C = rational curve in S, fully tangent to E at P.
- $\mathrm{Contr}_d(C) :=$ contribution of $d:1$ relative covers $C' \to C$ to $\mathcal{N}^P_{d[C]}(S,E)$.
- By GPS, $\mathrm{Contr}_d(C) = \dfrac{1}{d^2}\dbinom{d([C]\cdot E-1)-1}{d-1}$.
- Recall:

$$m^P_\beta = \sum_{\{k|\beta\,:\,P\in E(\beta/k)\}} \frac{(-1)^{(k-1)w/k}}{k^2}\mu(k)\mathcal{N}^P_{\beta/k}(S,E).$$

Define:

$$\mathrm{Contr}(C, m^P_{d[C]}) := \sum_{k|d} \frac{(-1)^{(k-1)d[C]\cdot E/k}}{k^2}\mu(k)\ \mathrm{Contr}_{d/k}(C),$$

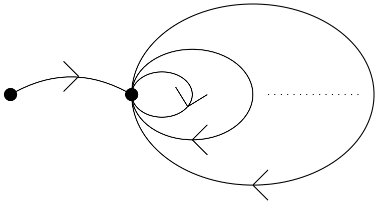
the contribution of covers of C to $m^P_{d[C]}$.

Multiple cover contributions are DT invariants

Theorem (relying on Reineke)

$$\mathrm{Contr}(C, m^P_{d[C]}) = DT_d^{([C]\cdot E-1)}.$$

Here, $DT_n^{(m)}$ is the degree n DT invariant of the m-loop quiver:



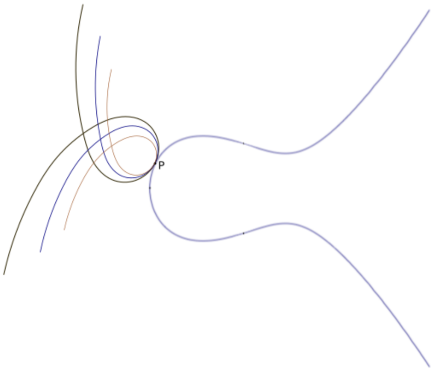
Question

What about relative covers over reducible curves?

β-primitivity

Definition

$P \in E$ is said to be β-primitive if $\beta|_E \sim wP$, but there is no decomposition into pseudo-effective curve classes $\beta = \beta' + \beta''$, with $\beta', \beta'' \not\sim 0$ and $\beta'|_E \sim w'P$.



Log BPS state counts

For a β-primitive P, set

$$M^P_{rel}(S,\beta) := \{ S \supset C \text{ rational } : [C] = \beta,\ C|_E = wP \text{ in one branch} \}.$$

Theorem (Enumerative interpretation)

$\exists$ explicit $m : M^P_{rel}(S,\beta) \to \mathbb{N}$ such that

$$m^P_\beta = \sum_{C \in M^P_{rel}(\beta)} m(C).$$

Moreover, $m(\text{nodal}) = 1$ and $m(\text{cusp}) = 2$.

Definition

For P a β-primitive point, the *log BPS numbers of degree* β are

$$m_\beta := m^P_\beta.$$

Conjecture B, restated

$$(-1)^{w-1}\frac{n_\beta}{w} = m_\beta.$$

Theorem

Conjecture B holds for all S and for classes β of arithmetic genus $p_a(\beta) := \frac{1}{2}\beta(\beta + K_S) + 1 \leq 2$.

Recall from the previous talk:

Theorem

$h = \#$ of (-1)-curves on S not intersecting β.

- *If $p_a(\beta) = 0$, then $n_\beta = (-1)^{w-1}w$.*
- *If $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $n_\beta = (-1)^{w-1}w(\chi(S) - h)$.*
- *If $\beta = -K_{S_8}$, then $n_\beta = (-1)^{w-1}w(\chi(S_8) + 1) = 12$.*
- *If $p_a(\beta) = 2$, then $n_\beta = (-1)^{w-1}w(\binom{\chi(S)-h}{2} + 5)$.*

Proposition

Assume $\exists$ irreducible reduced curve C of class β such that $C|_E \sim wP$ for some $P \in E$. Then there is a S.E.S.

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}_S(C-E)) \longrightarrow \mathrm{H}^0(\mathcal{O}_S(C)) \xrightarrow{\text{res}} \mathrm{H}^0(\mathcal{O}_E(wP)) \longrightarrow 0.$$

Moreover, $\dim\left(\mathrm{H}^0(\mathcal{O}_S(C-E))\right) = p_a(\beta)$.

Arithmetic genus 0

Assume that $p_a(\beta) = 0$. By the Proposition,

$$\mathrm{H}^0(\mathcal{O}_S(C)) \xrightarrow[\cong]{\text{res}} \mathrm{H}^0(\mathcal{O}_E(wP)).$$

Up to scalar multiple, there is exactly one section $s \in \mathrm{H}^0(\mathcal{O}_E(wP))$ vanishing at P with multiplicity w. Hence $m_\beta = 1$ as predicted.

Arithmetic genus 1

Assume that $p_a(\beta) = 1$ and that $\beta \neq -K_{S_8}$. By the Proposition,

$$H^0(\mathcal{O}_S(C)) \xrightarrow{\text{res}} H^0(\mathcal{O}_E(wP)),$$

with 1-dimensional kernel. Let $s \in H^0(\mathcal{O}_E(wP))$ be the only, up to $\mathbb{C}^*$, section vanishing at P with multiplicity w.

- Consider the linear system $|\,\text{res}^{-1}(\mathbb{C} \cdot s)|$.
- Step 1: Blow down all the (-1)-curves on S that do not intersect β.
- Step 2: Adding in E, we obtain a pencil Λ.
- There is one curve D in the pencil that is nodal at P.
- Step 3: Blow up the strict transform of P a number w of times.

Keeping track of the blowups

Exercise

We get the universal family $\mathcal{U} \to \mathbb{P}^1$ of the pencil and the proper transform of D is a cycle of w $\mathbb{P}^1$s.

- For a smooth curve C of the pencil, $\chi(C) = 0$.
- If C is nodal, $\chi(C) = 1$.
- If C is cuspidal, $\chi(C) = 2$.
- There are no worse singularities.

By the cut and paste properties of $\chi(-)$,

$$\chi(\mathcal{U}) = 0 \cdot \chi(\mathbb{P}^1) + \#\{\text{ nodal fibers }\} + 2 \cdot \#\{\text{ cuspidal fibers }\} + w.$$

On the other hand,

$$\chi(\mathcal{U}) = \chi(S) + \#\{\text{ blow ups }\} - \#\{\text{ blow downs}\}.$$

Hence,

$$m_\beta = \#\{\text{ nodal fibers }\} + 2 \cdot \#\{\text{ cuspidal fibers }\} = \chi(S) - h,$$

as predicted. Similarly for $\beta = -K_{S_8}$.

Arithmetic genus 2

Unfortunately waaaaaaay too technical for the time remaining. Hence...

Have a nice lunch and...

We hope you enjoyed our talks and...

Thank you!