Scattering diagram of $(\mathbb{P}^2, E)$ part II

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Based on work of

N. Takahashi, Carl-Pumperla-Siebert, Gräfnitz, 2×Bousseau

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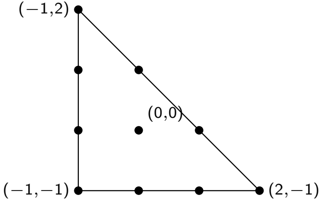
19 October 2020

A log smooth degeneration of $(\mathbb{P}^2, E)$

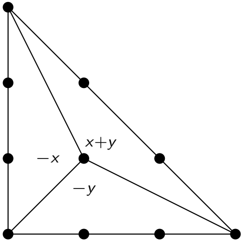
First steps

- Degenerate $\mathbb{P}^2 = \mathbb{P}_\Delta \rightsquigarrow \cup_i \mathbb{P}_{\Delta_i}$
- $\mathbb{P}_\Delta \supset E \rightsquigarrow D = \cup_i D_i \subset \cup_i \mathbb{P}_{\Delta_i}$, $D_i \subset \mathbb{P}_{\Delta_i}$
- $\cup_i \Delta_i$ is a regular polyhedral decomposition of Δ

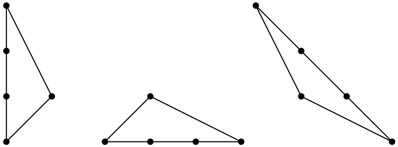
Let $M = \mathbb{Z}^2$ and consider the polytope Δ for $(\mathbb{P}^2, \mathcal{O}(3))$:



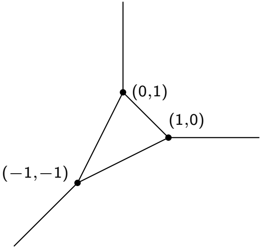
Take an integral regular subdivision of Δ, i.e. a subdivision induced as the "bending locus" of a piecewise linear convex function φ:



This determines a degeneration of $\mathbb{P}^2$ into the union of three weighted projective spaces $\mathbb{P}(1,1,3) = \mathbb{P}_{\Delta_i}$, where Δ_i is either of

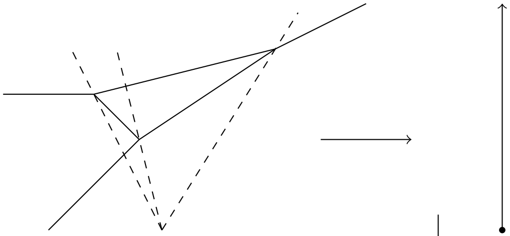


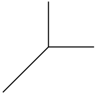
To describe the degeneration of $\mathbb{P}^2$ at the fan level, start with



The "local fans" at the vertices are the fans for the $\mathbb{P}(1,1,3)$'s.

Move the polyhedral decomposition of $\mathbb{R}^2$ to "height" 1 and take the cone over it, as well as the natural map to $\mathbb{A}^1$ given by "height":



The generic fiber is the *asymptotic fan*, in this case: 

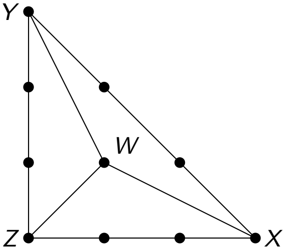
The components of the central fiber are given by the "quotient" by the vertical rays $\rightsquigarrow \mathbb{P}_{\Delta_i} \cong \mathbb{P}(1,1,3)$.

This is the desired degeneration $\mathbb{P}_\Delta \rightsquigarrow \cup_i \mathbb{P}_{\Delta_i}$. (More projectively, the total family can also be described from the dual non-compact polyhedron.)

In this degeneration the toric divisor of $\mathbb{P}^2$ goes to a cycle of lines, one in each of $\mathbb{P}_{\Delta_i}$.

Equations

Integral points of the polytope correspond to sections of ample line bundles of the family:



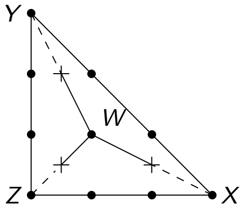
φ is defined by taking value 1 at X, Y, Z and value 0 at W. The smoothing parameter is $t = z^{(0,0,1)}$. This leads to the family

$$V\left(XYZ - t^3 W\right) \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^2$$

$V(W)$ is the toric horizontal divisor.

Gross-Siebert generalization

Integral points of the Gross-Siebert base correspond to sections of ample line bundles of the family:



We add a parameter s that pushes in the corners. g is a general cubic polynomial (e.g. $g = X^3 + Y^3 + Z^3$).

$$V\left(XYZ - t^3(W + sg)\right) \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^2$$

$V(W)$ is the horizontal divisor. It is a degeneration of a smooth cubic into a union of lines.

The family has 3 log singularities on smooth points of the common divisor of each of two $\mathbb{P}(1,1,3)$. Blowing them up we get a log smooth family.

Outline of proof of Gräfnitz

- Consider the just constructed log smooth degeneration of $(\mathbb{P}^2, D)$.
- Apply the decomposition formula of Abramovich-Chen-Gross-Siebert, indexed by tropical curves h in the dual intersection complex.
- Each term is a weighted count of curves that tropicalize to h.
- Compute the latter by applying GPS locally.
- Reformulate inside the scattering diagram, which is the dual intersection complex of the degeneration.

Conclusion

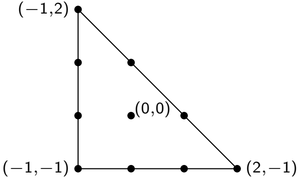
The wall-crossing functions encode $N^k_{0,d}$.

Relative/log Gromov-Witten invariants

- $P_d := 3d-$torsion points of E.
- For $p \in P_d$, $d(p) := \min\{k \mid p \in P_k\}$.
- $\overline{M}_0(\mathbb{P}^2/E, d) :=$ moduli space of degree d genus 0 basic stable log maps of maximal tangency (J. Li, Chen, Abramovich-Chen, Gross-Siebert)
- $\overline{M}_0(\mathbb{P}^2/E, d) = \bigsqcup_{p \in P_d} \overline{M}_0(\mathbb{P}^2/E, d)^p$
- $N_{0,d} := \int_{[\overline{M}_0(\mathbb{P}^2/E,d)]^{\mathrm{vir}}} 1 \in \mathbb{Q}$
- $N_{0,d}^p := \int_{[\overline{M}_0(\mathbb{P}^2/E,d)^p]^{\mathrm{vir}}} 1 \in \mathbb{Q}$
- $N_{0,d} = \sum_{p \in P_d} N_{0,d}^p$
- $N_{0,d}^p$ depends only on $d(p)$
- For $k|d$, $N_{0,d}^k := N_{0,d}^p$ for $p \in P_d$ with $d(p) = k$

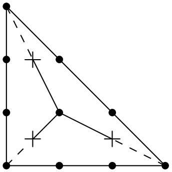
Gross-Siebert construction of dual intersection complex

Polytope of $(\mathbb{P}^2, \mathcal{O}(3))$

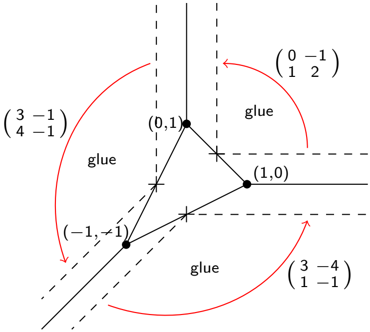


Smooth out boundary

by pushing singularities inside,
creating 3 focus-focus singularities

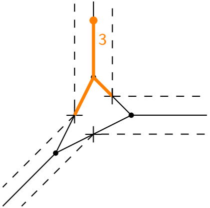


Dual intersection complex



The monodromy around each singularity is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

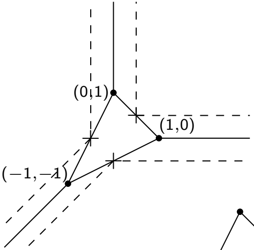
Each singularity has a monodromy-invariant line along which disks propagate.

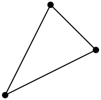


Disks satisfy the balancing condition:

Fundamental domain

In



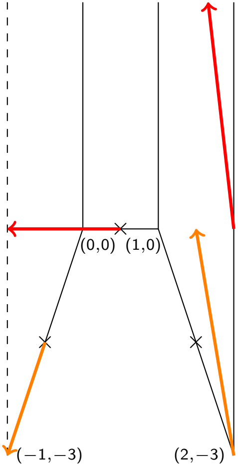
remove interior of  and glue.

Respect balancing condition at •,

only one infinite direction,

global monodromy is

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix}$$



Walls carry wall-crossing functions

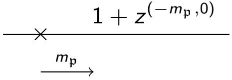
Away from the discriminant locus, B admits an integral affine structure $\Rightarrow$ sheaf of integral tangent vectors Λ_B.

$$\mathbb{C}[\Lambda_B \oplus \mathbb{Z}] := \langle z^{(m,a)} \mid m \in B, a \in \mathbb{Z} \rangle$$

$z^{(0,1)}$ is the smoothing parameter of the mirror family (to get the smoothing we need the multivalued piece-wise affine function φ, which we will continue to ignore).

The original wall-crossing function

Let $\mathfrak{p}$ be the wall directed by $m_{\mathfrak{p}} \in \Lambda_B$ coming out of a focus-focus singularity. Its wall-crossing function is



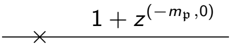
Proposition (Gräfnitz '20)

For a wall $(\mathfrak{p}, m_{\mathfrak{p}})$, denote by $\mathfrak{H}_{\mathfrak{p},w}$ the set of *tropical disks* ending on $\mathfrak{p}$ with final edge directed by $-wm_{\mathfrak{p}}$.

Each $h \in \mathfrak{H}_{\mathfrak{p},w}$ carries a combinatorial multiplicity N_h. Then

$$\log f_{\mathfrak{p}} = \sum_{w=1}^{\infty} \sum_{h \in \mathfrak{H}_{\mathfrak{p},w}} N_h \, z^{(-wm_{\mathfrak{p}},0)}$$

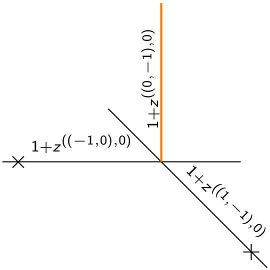
Example



$$\log\left(1 + z^{(-m_{\mathfrak{p}},0)}\right) = \sum_{w=1}^{\infty} \frac{(-1)^{w-1}}{w} z^{(-wm_{\mathfrak{p}},0)}$$

Simple scattering

If two initial walls $\mathfrak{p}, \mathfrak{p}'$ meet with $\det(m_{\mathfrak{p}}\ m_{\mathfrak{p}'}) = \pm 1$, then only one additional disk is produced directed by $m_{\mathfrak{p}} + m_{\mathfrak{p}'}$ and with wall-crossing function $1 + z^{(-m_{\mathfrak{p}} - m_{\mathfrak{p}'}, 0)}$:



If $\det(m_{\mathfrak{p}}\ m_{\mathfrak{p}'}) = \pm 2$, there is infinite scattering with rays directed by $n\, m_{\mathfrak{p}} + (n+1) m_{\mathfrak{p}'}$, $(n+1) m_{\mathfrak{p}} + n\, m_{\mathfrak{p}'}$ and $m_{\mathfrak{p}} + m_{\mathfrak{p}'}$.

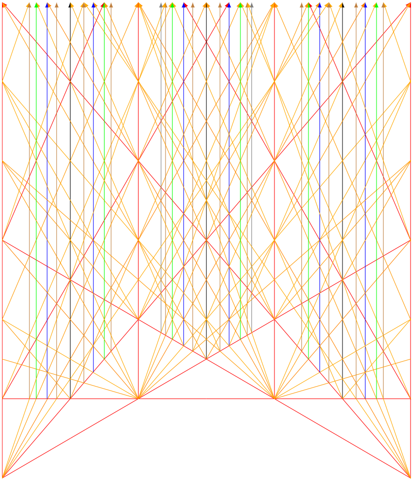
General scattering

If $\det(m_{\mathfrak{p}}\ m_{\mathfrak{p}'}) = \pm 3$, there are two regions with countably infinite scattering and a region where all rays with rational slope occur.

In general (also with more than 2 incoming rays), the scattering algorithm is described by GPS with coefficients given as log GW invs.

It works $z^{(0,1)}$-order by $z^{(0,1)}$-order.

Consistency condition needs to be satisfied at each joint.



- The circle at ∞ is $\mathbb{R}/3\mathbb{Z}$ with the vertical rays ending in $\mathbb{Q}/3\mathbb{Z}$.
- G abelian group, $x \in G$ with $3 \mid \mathrm{ord}(x)$, $d(x) := \min\{d \mid 3d\,x = 0\}$
- $r_\ell := \#\{x \in \mathbb{Z}/3\ell \mid d(x) = \ell\}$
- $s_{k,\ell} := \#\{(a,b) \in \mathbb{Z}/3k \times \mathbb{Z}/3k \mid d((a,b)) = k,\ d(a) = \ell\}$

$x \in \mathbb{Q}/3\mathbb{Z}$, $d(x) = \ell$, then

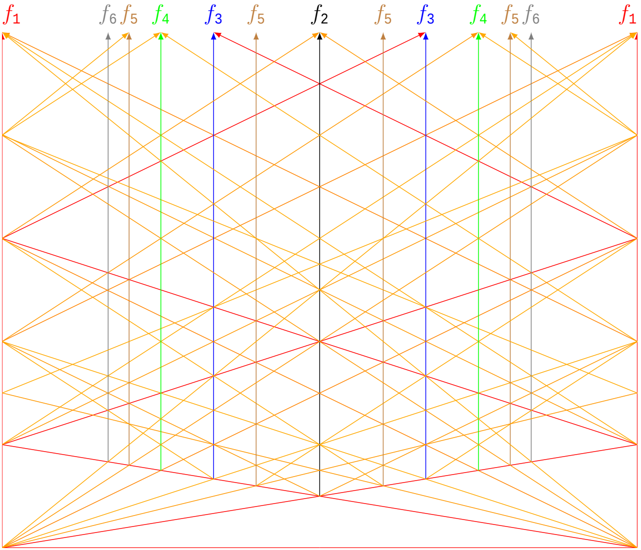
$$\frac{s_{k,\ell}}{r_\ell} = \#\{p \in P_d \mid d(p) = k,\ p \text{ specializes to } x\}$$

Theorem (Gräfnitz '20)

For $x = z^{((0,-1),0)}$ and $\mathfrak{p}$ an unbounded wall of order ℓ

$$\log f_{\mathfrak{p}} = \sum_{d=1}^{\infty} 3d \left(\sum_{\ell | k | d} \frac{s_{k,\ell}}{r_\ell} N_{0,d}^{k} \right) x^{3d}.$$

Zoom into the central third (image from Gräfnitz)



Also from Gräfnitz

The sage code gives the following:

$$\log f_1 = 9x^3 + \frac{63}{2}x^6 + 246x^9 + \frac{9279}{4}x^{12} + \frac{175464}{5}x^{15} + 307041x^{18} + O(x^{21})$$

$$\log f_2 = 36x^6 + 2322x^{12} + 307164x^{18} + O(x^{21})$$

$$\log f_3 = 243x^9 + \frac{614061}{2}x^{18} + O(x^{21})$$

$$\log f_4 = 2304x^{12} + O(x^{21})$$

$$\log f_5 = 25425x^{15} + O(x^{21})$$

$$\log f_6 = 307152x^{18} + O(x^{21})$$

Insight of Bousseau

Using

$$\log f_{\mathfrak{p}} = \sum_{d=1}^{\infty} 3d \left(\sum_{\ell | k | d} \frac{s_{k,\ell}}{r_\ell} N_{0,d}^{k} \right) x^{3d},$$

how do you prove that the BPS invariant $\Omega_{d,k}^{\mathbb{P}^2/E}$ is independent of k? Here

$$(-1)^{d-1} N_{0,d}^{k} = \sum_{k|d'|d} \frac{1}{(d/d')^2} \Omega_{d',k}^{\mathbb{P}^2/E}.$$

Idea

Translate the question to a question about DT invariants of $K_{\mathbb{P}^2}$.

Heuristics I

Heuristics II

Heuristics III

$$U := \{(x,y)\in\mathbb{R}^2 \,\big|\, y > -\frac{x^2}{2}\}$$

$$K_0(\mathbb{P}^2) := \Gamma \simeq \mathbb{Z}^3 \ni (r(F), d(F), \chi(F))$$

$$U \hookrightarrow \mathrm{Stab}\,\mathrm{D}^b(\mathbb{P}^2), \quad \sigma \mapsto (\mathcal{A}^\sigma, Z^\sigma)$$

Central charge $Z^\sigma : \Gamma \to \mathbb{C}, \quad \gamma \mapsto Z^\sigma_\gamma$

$F \in \mathcal{A}^\sigma$, phase $\phi(F) := \frac{1}{\pi}\arg Z^\sigma_{\gamma(F)}$

$$\sigma \in U, \gamma \in \Gamma \rightsquigarrow M^\sigma_\gamma := \{F \,\big|\, \sigma\text{-semistable}, \gamma(F) = \gamma\}/_{S\text{-equivalence}}$$

Wall and chamber structure on U, M^σ_γ jumps along walls. For $y >> 0$, (x,y)-semistability $\rightsquigarrow$ Gieseker-semistability.

$$\Omega^\sigma_\gamma := (-1)^{\dim M^\sigma_\gamma} Ie(M^\sigma_\gamma) \in \mathbb{Z}$$

satisfy Kontsevich-Soibelman wall-crossing.

Wall-structure

For $\gamma \in \Gamma$, rays

$$R_\gamma := \{\sigma \in U \,|\, \mathrm{Re}\, Z_\gamma^\sigma = 0,\ \Omega_\gamma^\sigma \neq 0\}$$

$\rightsquigarrow$ straight line in direction $(-r, d)$. R_γ decomposes into segments, where the $\Omega_{k\gamma}^\sigma$ do not jump. To these we attach the function

$$\sum_{k \geq 1} \frac{\Omega_{k\gamma}^\sigma}{k^2} z^{(kr, -kd)}.$$

Theorem (Bousseau '19)

This defines a consistent scattering diagram.

Comparing the two scattering diagrams

Theorem (Bousseau '19)

Both scattering diagrams have the same initial conditions $\Rightarrow$ they are the same.

Going to ∞

Theorem (Bousseau '19)

The function attached to unbounded vertical rays with $\gamma = (0, d, \chi)$ is

$$(-1)^{d-1}\left(\sum_{\ell|(d,\chi)} \frac{1}{\ell^2}\Omega^{\mathbb{P}^2}_{\frac{d}{\ell},\frac{\chi}{\ell}}\right) z^{(0,-d)}$$

Corollary

Let $\ell_{d,\chi} := \frac{d}{(d,\chi)}$. Then

$$(-1)^{d-1}\sum_{\ell|(d,\chi)} \frac{1}{\ell^2}\Omega^{\mathbb{P}^2}_{\frac{d}{\ell},\frac{\chi}{\ell}} = \sum_{\ell_{d,\chi}|k|d} \frac{s_{k,\ell_{d,\chi}}}{r_{\ell_{d,\chi}}} N^k_{0,d}$$

Moreover,

$$\sum_{\ell|(d,\chi)} \frac{1}{\ell^2}\Omega^{\mathbb{P}^2}_{\frac{d}{\ell},\frac{\chi}{\ell}} = \sum_{\ell_{d,\chi}|d'|d} \frac{1}{(d/d')^2}\Omega^{\mathbb{P}^2}_{d',\chi\frac{d'}{d}}$$

Hence

$$\sum_{\ell_{d,\chi}|d'|d} \frac{1}{(d/d')^2} \Omega^{\mathbb{P}^2}_{d',\chi\frac{d'}{d}} = \sum_{\ell_{d,\chi}|k|d} \frac{s_{k,\ell_{d,\chi}}}{r_{\ell_{d,\chi}}} (-1)^{d-1} N^k_{0,d}$$

Corollary (Bousseau '19)

$$\Omega^{\mathbb{P}^2}_{d,\chi} = \sum_{\ell_{d,\chi}|k|d} \frac{s_{k,\ell_{d,\chi}}}{r_{\ell_{d,\chi}}} \Omega^{\mathbb{P}^2/E}_{d,k}$$

Conclude by using

Theorem (Bousseau '19)

$\Omega^{\mathbb{P}^2}_{d,\chi}$ is independent of χ.

Thank you